Chapter 2 Problems

1) \( g = C \sqrt{N} \)

\[ \sigma = \log g = \log + \frac{3}{2} N \log u \]

a) \( \frac{1}{\varepsilon} = \frac{3}{8 \pi} = \frac{3}{2} \cdot \frac{N}{u} \), whence

\[ u = \frac{3}{2} N \varepsilon \]

b) \( \frac{\partial^2 \sigma}{\partial u^2} \bigg|_N = -\frac{3}{2} \frac{N}{u^2} < 0 \)

Note that \( \frac{\partial^2 \sigma}{\partial u^2} \bigg|_N = \partial u \left( \frac{1}{\varepsilon} \right) \bigg|_N = -\frac{1}{\varepsilon^2} \frac{\partial^2 \varepsilon}{\partial u} \bigg|_N \)

\[ = -\frac{3}{8 \pi} \frac{\partial^2 \varepsilon}{\partial u} \bigg|_1 \]

Hence the heat capacity is \( > 0 \).

2) If \( \sigma(s) = \log(g_0) - \frac{2 s^2}{N} \)

and \( u = -\langle s \rangle 2 m b \), then at equilibrium

\[ \sigma(u) = \log(g_0) - \frac{2}{N} \left( \frac{u}{2 m b} \right)^2 \]

Hence \( \frac{1}{\varepsilon} = \frac{\partial \sigma}{\partial u} \bigg|_b = -\frac{u}{N m^2 b^2} \)

\[ = \frac{\langle s \rangle 2 m b}{N m^2 b^2} = \left( \frac{2 \langle s \rangle}{N} \right) \frac{1}{m b} \]

\( \varepsilon = \frac{2 \langle s \rangle}{N} \cdot m b^{1/2} \)
a) \[ g(N,n) = \frac{(N+n-1)!}{n!(N-1)!} \]

Since \[ \log m! = m \log m - m \]

\[ \log(N+n-1)! = (N+n-1)\log(N+n-1) - (N+n-1) \]

\[ = (N+n)\log(N+n) - (N+n) \]

\[ \log g(N) = (N+n)\log(N+n) - (N+n) \]

\[ = (N+n)\log(N+n) - n \log n - N \log N \]

\[ = N \log \left( \frac{N+n}{n} \right) + n \sum \log \left( \frac{N+n}{n} \right) \]

b) Since \[ u = n+kw \]

\[ \frac{1}{z} = \frac{3z}{5u} = \frac{3}{5} \frac{3z}{5u} \]

\[ \frac{kw}{z} = \int N \left[ \frac{1}{N+n} + \log \left( \frac{N+n}{n} \right) \right] + \]

\[ n \left[ -\frac{1}{n} + \frac{1}{N+n} \right] = \log \left( \frac{N+n}{n} \right) \]
Hence, \( \frac{N+n}{n} = \frac{N}{n} + 1 = e^{\frac{n}{N}} \)

and \( n = \frac{U}{Kw} = N \sum e^{-\frac{x}{kT}} - 1 \) 

a) Since the choice of each letter has 44 possibilities, a specific sequence has probability \( \left(\frac{44}{44}\right)^n \), where \( n \) is the number of characters in a sequence. Since \( 44 = 10^{1.64} \), we have \( \left(\frac{44}{44}\right)^{100,000} = 10^{-164.345} \)

b) Here’s one answer: \( 10^{10} \) monkeys generate \( 10 \times 10^{11} \) characters/s. One monkey generates the necessary 15 characters in \( 10^{48} \) s. In \( 10^{18} \) s, one monkey generates \( 10^{44} \) sequences. \( 10^{10} \) monkeys generate \( 10^{10} \times 10^{44} = 10^{54} \) sequences.

We ask how many ways can we draw \( m=10^{28} \) objects from a collection of \( n=10^{169.345} \) objects such that at least one of the objects is Hamlet = H. Note that we can draw H more than once. Thus, there are \( N^m \) ways of making the draws. If we remove H from the collection, there are \( (N-1)^m \) ways of making the draws, i.e. the number of
ways of drawing m objects from a collection of N objects, such that H never appears. Thus \( N^m - (N-1)^m \) is the number of draws such that H appears at least once. The probability of obtaining H at least once is

\[
P(H) = \frac{N^m - (N-1)^m}{N^m} = 1 - \left(1 - \frac{1}{N}\right)^m
\]

If we write \( m = \alpha N \), then

\[
P(H) = 1 - \left(1 - \frac{1}{N}\right)^\alpha = 1 - \hat{r}^\alpha
\]

In our case \( \alpha = \frac{10^{28}}{10^{164,345}} \), and to lowest order

\[
P(H) \approx \alpha
\]
5) \[ g_1 g_2 = g_0 g_0 \exp \left\{ - \frac{2(N_1 + N_2)}{N_1 N_2} (s_1 - x s)^2 - \frac{2S^2}{N_2} \right\} \]

Since we take \( s = 0 \), \( N_1 = N_2 \), \( x = \frac{N_1}{N_1 + N_2} \)

\[ g_1 g_2 = 0 \]

a) For \( N_1 = N_2 = 10^{22} \)

\[ x = \frac{x_1 + 10^{11}}{x_1 + x_2} = 0, \quad s_1 = 10^{11} \]

\[ 4s^2/N_1 = 4 \times 10^{22} / 10^{22} = 4 \]

Hence, \( g_1 g_2 \) for \( s_1 = x_1 + 10^{11} \) is down from the \( s_1 \) value by

\[ e = 0.018 \]

b) (and c)

For one value of \( s_1 \)

\[ g_1 g_2 = 0 \]

In fact, we can give values of \( s_1 \) with

\[ s_1 + s_2 = s \]

had vary over a substantial range. However, we know that if

\[ + \frac{2}{N_2 x} (s_1 - x s)^2 = + \frac{4}{N_2} (s_1 - x s)^2 = \frac{14}{N_1} (s_1 - x s)^2 \]
deviates much above 1, the contribution to a sum

\[ g_T = \frac{1}{s_1} \sum_{s_1} g_1(N_1, s_1) g_2(N_2, s-s_1) \]

will be small.

If we want the total sum above overall \( s_1 \), we can write

\[ g_T = \int_{s_1}^{s_1 \text{max}} \exp \left[ -\frac{2}{N_2 x} (s_1 - xs)^2 - \frac{2s^2}{N_2} (1-x) \right] ds_1 \]

Note that \( s_1 + s_2 = s \), but also \( s_1 \) can be negative, and very large in magnitude. Hence, we can take the limits on \( s_1 \) to \( \pm \infty \),

\[ g_T = g_T^{\text{max}} \int_{-\infty}^{\infty} e^{-\frac{2s^2}{N_2 x}} ds_1 = g_T^{\text{max}} \int_{-\infty}^{\infty} e^{-2s^2/(N_2 x)} ds_1 \]

We show use the fact that

\[ \int_{-\infty}^{\infty} e^{-ax^2} dx = (\pi a)^{1/2} \]

\[ \Rightarrow g_T = \left( \frac{\pi N_2 x}{2} \right)^{1/2} g_T^{\text{max}} \]
Note that \((\frac{N-Ak}{2})^2\) is large, but if we now allow a relatively small range of \(s_1\) is around \(x^2 = \frac{\xi}{s_1}\),

\[
g_{\text{max}} = g_{10} g_{z_0} e^{-\frac{2\xi^2}{N_2^2} (1-x)} \left[ \int_{s_{\text{max}}}^{\infty} \frac{g}{s^2} ds - \frac{g}{s_{\text{max}}^2} \right]
\]

where \(s_{\text{max}} \propto N_1\), and a only moderately long, be slightly longer than 1,

\[
g_{\text{max}} = g_{10} g_{z_0} e^{-\frac{2\xi^2}{N_2^2} \int_{a}^{\infty} \frac{g}{s^2} ds - \int_{a}^{\infty} \frac{g}{s^2} ds}
\]

Then

\[
\frac{g_{\text{max}}}{g_{10}} = \frac{\int_{a}^{\infty} e^{-x^2} dx - \int_{a}^{\infty} e^{-x^2} dx}{\int_{0}^{\infty} e^{-x^2} dx}
\]

which very quickly approaches 1.

For \(a = \frac{g_{\text{max}}}{g_{10}} =
\begin{array}{cc}
1 & 0.84 \\
2 & 0.995 \\
2.5 & 0.9996
\end{array}