Green’s Function Measurements in 2D Granular Materials

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Abstract

In this paper, we study how forces propagate in granular systems under various conditions. We describe experiments measuring an ensemble of responses for 2D granular systems consisting of photoelastic particles. We obtained results, including ensemble-averaged responses, for four types of particle packing geometry: bidisperse packings of disks with different amount of disorder, disks packed in a regular rectangular lattice, systems with forces applied at an arbitrary angle at the surface, and systems with shear deformation, hence with a textured or anisotropic feature. We experimentally show that disorder, packing structure, friction and texture can significantly affect the average force response in granular systems. We also explore the spatial properties of forces chains in a textured system. In this case, the spatial correlation function has a range of only one particle size in the direction transverse to the chains, and varies as a power law in the direction of the chains, with an exponent of -0.81.
I. INTRODUCTION

Force propagation in granular materials is a fundamental, but unresolved problem [1,2] which has received much attention recently [3–7]. At the microscopic level, an applied external force results in a filamentary network of stress/force chains [8], where a small fraction of the total number of grains carry the majority of the force. We show an example of these chains in Fig. 19. In general, predictability of grain-scale details of force propagation is limited by the indeterminacy of Coulomb friction at grain and wall contacts [9]. However, through the process of averaging over multiple realizations, it is possible to obtain a good statistical description of granular materials.

A number of substantially different models exist to describe various aspects of force propagation in dense granular materials [10]. For the continuum case, we emphasize the dramatic range of predictions by noting that these models predict qualitatively different partial differential equations (PDEs) for the stress propagation in a granular system. For example, classical elastoplastic models [11], are described by elliptic equations below the plastic threshold (which is the region we consider here) or hyperbolic above the plastic threshold. The continuum limit of the q model of Coppersmith et al. [12] is a parabolic PDE. And, the oriented stress linearity (OSL) model [13] of Bouchaud et al. uses a hyperbolic PDE in the absence of randomness.

A useful tool for distinguishing among these models is the Green’s function, i.e. the stress response function for a localized force [2]. In previous work [3,4], we have investigated the simplest case: responses to a small force applied at the boundary of 3D and 2D systems, respectively. We showed that spatial ordering of the particles is a key factor in the force response: ordered packings have a response with a strong propagative part in 2D and strong disordered packings show an elastic-like response both in 2D and 3D.

Very recently, several competing models have been proposed to account for the experimental findings. Among them, the force chain splitting model or double-Y model by Bouchaud et al. [5,?] is a Boltzmann equation for the probability density of finding a
force chain with certain intensity at a given direction. In the presence of strong disorder and isotropic “scattering” of force chains, the authors derive stress equations formally identical to those of classical elasticity. An alternative model by Goldenberg and Goldhirsch [6] assumes that nearest neighbors in a 2D packing of disks are coupled by uni-directional linear springs. These authors propose that the experimental results can be described using anisotropic elasticity, leading to an elliptic type PDE in the continuum limit. Both models seem to reasonably explain the experiments, however, it is thus far not clear which mechanism, hyperbolicity or elasticity, is more fundamentally a part of force propagation in a granular material.

Another important aspect of the problem concerns textures in a granular system. A texture can be loosely defined as a non-isotropic distribution of contacts between grain [5], or more rigorously, as a second order texture or fabric tensor as defined in, for example, Ref. [2]. Experiments [20,21] have shown that the existence of textures due to different deposition procedures can determine the way forces are transmitted and produce different stress distributions in granular materials. Among recent models, the force chain splitting model emphasizes the need to incorporate a texture [5], and the spring model by Goldenberg and Goldhirsch [6] explores the possibility of anisotropic elasticity to account for the hyperbolic features observed in experiments.

This paper explores through experiments the issues of the local response of granular materials to small forces. This work is also a substantial extension of recently published work [4]. We explore the regime of limited deformation, so that to a reasonable approximation, our results represent a true Green’s function. We present information on the methods used and we explore a broad range of systems. In particular, we consider: 1) responses of bidisperse systems with different amount of disorder, 2) responses of rectangular packings of disks, 3) responses to forces applied at arbitrary angles to the surface, and 4) responses of a sheared, thus textured/anisotropic, system.

The organization of this work is as follows. In Section II, we describe the experimental procedures, methods and issues common to all experiments. In Section III, we describe four
experiments that address the four points above. Finally, we draw conclusions in Section IV.

II. COMMON FEATURES

In this section, we first describe experimental procedures and analysis methods common to all experiments. We then address the issues of fluctuations and linearity in our measurements.

A. Experimental procedures

1. Experimental arrangements

Photoelastic measurements [15] in 2D provide a unique opportunity to obtain information about the internal structure of granular materials. The experiments we describe below typically use a layer of photoelastic grains (i.e. birefringent under strain) consisting of either disks or pentagonal particles. All the particles are cut from flat sheets of a commercially available material (Measurements Group, Inc.) with a Young’s modulus of 4 MPa and a Poisson ratio of 0.5.

The grains are usually contained between two transparent Plexiglas plates and this “sandwich” is placed between a pair of left and right hand circular polarizers as shown in Fig. 1a. For illumination, we use a light box such as that used to read x-ray films, because this provides a relatively homogeneous source. We record these with a digital camera at a resolution of 640 × 480 pixels. Fig. 2 shows a typical intensity picture for a disk under diametrical compression from experiments, as well as a similar pattern from numerical calculations.

In general, we can either perform the experiments in a vertical plane or a horizontal plane. When we use the former arrangement, the effect of hydrostatic head must be removed. We note that most experiments were performed with a vertical orientation because there is then minimal friction with the plates that contain the particles.
2. Force measurement

A key issue is how to deduce forces on a particle, i.e. forces at a grain scale. To understand this technique, we note that when the photoelastic grains are subjected to stresses, they become birefringent. When light travels through the particles along the direction normal to the plane of the experiment, the emerging light intensity $I$ is a function of position $(x, y)$ in the plane of the experiment and of the applied load, due to stress-induced birefringence, Specifically, the local light intensity is given by

$$I(x, y) = I_0 \sin^2[(\sigma_2 - \text{sigma}_1)tC/\lambda]$$

(1)

where the $\sigma_i(x, y)$ are the principle stresses, $t$ is the thickness of the sample, $C$ is the stress optic coefficient, and $\lambda$ is the wavelength of the radiation. In typical photoelastic images of the particles, bands corresponding to different stresses occur, where neighboring bands are separated by a phase difference of $\pi$ in the argument of the sine function above.

In general, the complete inverse problem that extracts vector forces on a particle for a given photoelastic image is a formidable problem. In these experiments, we use an empirical approach that allows us to obtain force at the grain scale with reasonable accuracy and much more simply than a complete calculation. The basis of this process is the fact that as the applied force at contact increases, the number of fringes (black or white bands) also increases monotonically. We exploit this fact to produce a force calibration in terms of quantity which we denote by $G^2$:

$$G^2 \equiv |\nabla I|^2 = \left[\frac{(I_{i-1,j} - I_{i+1,j})^2}{2} + \frac{(I_{i,j-1} - I_{i,j+1})^2}{2} + \frac{(I_{i,j+1} - I_{i+1,j+1})^2}{2}\right]/4,$$

(2)

where the $I_{i,j}$ is the intensity of the pixel $(i,j)$ as shown in Fig. 1c. The indices $i, j$ are the discrete replacement of the continuous variables $x, y$. Note that to avoid directional preference, the vertical, horizontal, and both diagonal gradients are squared and averaged. We compute $G^2(i, j)$ for each pixel $(i, j)$. 

5
For a particle or collection of particles covered by \( N \) pixels we calculate the average square gradients: 
\[
\langle G^2 \rangle = \frac{1}{N} \sum_{k=1}^{N} |\nabla I_k|^2.
\]
As the number of fringes increase, so does this average square gradient. The method can be applied to calibrate the mean force on a single particle or a larger assembly of particles. We obtained calibrations by either: (1) applying known forces to the boundary of a small number of particles and at the same time measuring \( \langle G^2 \rangle \), or (2) by applying various uniform loads to the upper surface of a large rectangular sample (width larger than height to avoid the Janssen effect) as shown in Fig. 3b. We show here a calibration curve, Fig. 3(a), using the second method. The validity of the \( G^2 \) calibration method is tested by measuring the hydrostatic pressure vs. depth \( z \). In a static system without external load, the hydrostatic pressure due to gravity is: 
\[
P = \gamma \rho g z,
\]
where \( \rho \) is the density of the material, \( g \) is gravity and \( \gamma \) is the packing fraction of the particles which is unity for a solid or liquid. In the experiment, the density, \( \rho \), is \( 1.15 \times 10^3 Kg \cdot m^{-3} \) and the typical packing fraction \( \gamma \) is \( \approx 0.75 \) for pentagons and 0.91 for triangular packed disks. The comparison between the expected and experimental data obtained from the \( G^2 \) method is shown in Fig. 3b and we see they agree well. These calibrations are effective until the forces are so large that it is no longer possible to resolve the fringes on a particle clearly, a condition that does not occur in these studies.

3. Procedures

The typical procedure was as follows. The particles were first placed in the apparatus. We then obtained a sequence of images. The first two images, made in the absence of the applied load, yielded the particle locations and the background photoelastic image. The first type of image were taken without the polarizers in place, and from these, we extracted particle positions and contacts. We then measured the system point-force response by placing a known weight carefully on top of one particle at the surface or by using a high precision digital force gauge (model DPS-110 from Imada Inc.) to produce a local force. With the local applied force in place, we obtained a photoelastic image. We then removed
the local force and obtained one last image without polarizers to ascertain if there had been any particle movement. In general, we did not use trials in which there were changes in the particle packing after the local force was removed. By computing the $G^2$ at the pixel scale for each image and subtracting the background from the response with load, we obtained the stress difference between successive images of $G^2$, containing only the response from the point perturbation. We refer to this difference as $\Delta G^2$, or as $G^2$ in the cases where no confusion is caused. Only responses where the particles remained undisturbed by this process were used.

The responses typically differ significantly from realization to realization. This is true even for the case of nearly regular grain packings, since the frictional forces at the contacts are determined by the microscopic details of the preparation history, something that in generally is not known. Hence, it is necessary to develop an ensemble of measurements in order to extract the mean behavior. In order to obtain an ensemble-average response, we repeated measurements on a given system for many different rearrangements of the particles, typically 50 times for each set of data. We will discuss more on this later. For responses in each realization, we denote the stress at the pixel scale as $G^2(x, y, n)$, where $n$ represents the realization number. We average the mean response in two ways. First, we compute the average of $G^2(x, y, n)$ over $n$. We then carry out a coarse-graining at the scale of a single particle, since variations in $G^2$ below the particle size are not meaningful here. The result is donoted by $\overline{G^2(x, y, n)}$.

**B. Fluctuations Among Trials**

We first consider fluctuations of the stress response from realization to realization. We note that this issue has also been addressed in recent theoretical studies by Claudin et al. [14]. As noted above, any given realization showed significantly different response at the grain scale from others in its class. At the same time, the ensemble contains important information concerning the range of results that might be encountered on any given realization. To
characterize the fluctuations from one realization to another, we calculate the standard deviation of the stress for each position: $$rms(x, y) = \sqrt{Var}$$, where

$$Var = \frac{1}{N-1} \sum_{n=1}^{N} (G^2(x, y, n) - \overline{G^2(x, y)})^2.$$  

(3)

As an example, we show the resulting $$rms(x, y)$$ data for a hexagonal disk packing in Fig. 4, using a greyscale representation, where brighter regions represent larger values of the $$rms$$. We see that the $$rms$$ image has a similar shape to the mean image. The similarity in the spatial pattern associated with the mean and the $$rms$$ suggests that there is a simple point-wise relation between these two quantities. We explore this by determining the distribution of $$rms(x, y)/\overline{G^2(x, y)}$$ for all points in the system. These data are given by the solid circles in Fig. 4b. The distribution has a narrow peak at around 2, indicating a large percentage of the distributions with a $$rms$$ to mean ratio close to 2.

The rest of the curves in Fig. 4b are similar plots for bidisperse systems of disks with different amount of disorder, and pentagonal systems, which we will discuss in next section. Briefly, the amount of spatial disorder increases as we change the system from monodisperse disks to bi-disperse disks and finally to pentagons. When the amount of disorder increases, we find fluctuations become stronger by noting the peaks in the distributions of $$rms/\overline{G^2(x, y)}$$ shift to larger values, and the distributions become wider.

C. Linearity of the Response

Additional issues that are of considerable importance in all the measurements are that of reversibility and of linearity. The first of these refers to the fact that the particles return to their unperturbed state after the applied local force is removed. The second addresses the issue of the functional relationship between the size of the applied force and the response at a given point. With the exception of bidisperse systems of disks, for the measurements reported here, deformations are reversible. However, that reversibility does not necessarily mean linearity.
For both ordered triangular disk packings and disordered pentagonal packings, when the applied force is below about 0.5 N, there is a good linearity within the error bars as shown in Fig. 5. In order to optimize the signal to noise ratio, yet avoid nonlinear effects, we usually chose a working force close to the upper bound of this linear region.

III. EXPERIMENTAL RESULTS

In this section, we describe four experiments that address various factors affecting the force propagation in granular systems, including disorder, packing structure, friction, direction of applied forces, and textures.

A. Role of packing disorder: Responses of bidisperse systems

Disorder at particle contacts may arise from at least two sources. One is the presence of geometrical disorder in the packing. The other is the random disorder in the contact forces, due for instance, to frictional indeterminacy. We first consider the effects of disorder on the packing.

The first way that we do so is by determining the force response for bidisperse systems with varying amounts of disorder. We modify the amount of disorder in a controlled way that we characterize by the parameter $\mathcal{A}$ of Luding et al. [16,4] who investigated ordering in granular gases. The quantity $\mathcal{A}$ is a dimensionless measure of the width of the disk size distribution. The value $\mathcal{A} = 1$ corresponds to perfect order in a monodisperse situation, and the deviation from unity is proportional to the degree of poly-dispersity or disorder in the system. In a bi-disperse system, if we denote the respective radii of smaller and larger particles as $a_1$ and $a_2$, the corresponding particle numbers as $N_1$ and $N_2$, the size ratio as $R = a_1/a_2$, and the number fractions as $n_i = N_i/N$ ($i=1,2$) with the total number of particles $N = N_1 + N_2$, then the parameter $\mathcal{A}$ is:

$$\mathcal{A} = \frac{\langle a \rangle^2}{\langle a^2 \rangle} = \frac{[n_1 + (1 - n_1)/R]^2}{n_1 + (1 - n_1)/R^2},$$

(4)
More generally, for polydisperse systems characterized by a radius distribution \( w(a) \), the average quantities such as \( \langle a^m \rangle \) are given by \( \langle a^m \rangle = \int w(a) a^m \, da / \int w(a) \, da \).

We prepared each sample by mixing about 500 small and 500 large disks in a container, so that \( n_1 \approx n_2 \approx 0.5 \). We then randomly chose one particle to add to the upper surface of the sample until the full amount of particles was in place. The exact values of \( n_1 \) and \( n_2 \) were determined later from images showing particle configurations based on particle identification software mentioned above.

The experiments on bidisperse systems were performed in a vertical plane [4]. Besides the \( \mathcal{A} = 1 \) case for ordered monodisperse packing, we have used disks with 3 different diameters and obtained bidisperse systems with three different \( \mathcal{A} \) values, i.e. \( \mathcal{A} = 0.993, 0.988 \) and 0.965. Based on Luding et al.'s phase diagram for granular gases [17], it seems likely that the \( \mathcal{A} = 0.965 \) case is already in the disordered region.

An alternative method to quantify the disorder of the system is to calculate the particle-particle positional autocorrelation function. The autocorrelation function is defined as [16],

\[
g(r) = \frac{2A}{N(N-1)} v_r \sum_{i=1}^{N} \sum_{j=1}^{i-1} \theta(r_{ij} - r)\theta(r + \Delta r - r_{ij}),
\]

where \( A \) is the area of the system, \( N \) the number of particles, and \( A_r = \pi(2r + \Delta r)\Delta r \) the area of a ring between \( r \) to \( r + \Delta r \), \( r_{ij} \) the distance between particle \( i \) and \( j \). The two \( \theta \) functions select all particle pairs with distances between \( r \) and \( r + \Delta r \). For bidisperse systems, we calculate the autocorrelation function between both the same species and corresponding correlation function between different species. When calculating the autocorrelation function for different species, the weight \( N(N-1)/2 \) in the above equation needs to be changed to \( N_1N_2 \) and the indices \( i \) and \( j \) run from 1 to \( N_1 \) and \( N_2 \) respectively, in order to account for all pairs of different kinds. Here, \( N_1 \) and \( N_2 \) denote the number of two species of particles.

In Fig. 9, we show particle-particle correlation functions for the monodisperse and three bidisperse disk systems that we have investigated. In Fig. 9a, we contrast the correlation functions calculated both from a perfect triangular disk lattice and a triangular lattice from the experiments. The correlation for the triangular lattice from experiments is apparently
not perfect. However, both correlation functions show a long range order with peaks at 1, $\sqrt{3}$, 2, ..., indicating a triangular lattice. In Fig. 9b, c and d, correlation functions are calculated for the same species and for mixtures of different disk species. In contrast to the monodisperse case, correlations of bidisperse systems decrease very quickly to 1 over a distance of several disk diameters, indicating increasing disorder as the control parameter $\mathcal{A}$ decreases.

By increasing the amount of disorder in the system, as measured by $\mathcal{A}$, we observed responses that change from a two-peak structure to a response very similar to that of a disordered pentagonal system. In Fig. 6, we give a greyscale representation of the average response to a point force for three bidisperse systems with $\mathcal{A} = 0.993$, 0.988 and 0.965, respectively. In Fig. 7, we present the same results by showing the response along a series of horizontal lines at a number of depths, $z$, measured from the source. For the largest $\mathcal{A}$, $\mathcal{A} = 0.993$, Fig. 6a and b show responses with two-peak features that resemble the response structure for ordered monodisperse disks. However, with decreasing $\mathcal{A}$, the propagative features become progressively weaker. In Fig. 7c, the two-peak feature has completely disappeared, and the response is similar to that of a disordered pentagonal system, presented below. The change from a two-peak to a one-peak structure presents clear evidence of the important role of disorder in force responses.

It is useful to compare these observations to the prediction by Claudin et al. [14] In the case of weak disorder, these authors predict a Convection-Diffusion equation, as discussed in the Appendix. This equation is characterized by a propagation speed, $c$ and a diffusivity, $D$. In order to estimate how these coefficients $c$ change when the amount of disorder changes in the systems, we extracted $c$ and $D$ by nonlinear least-squares fits of the mean responses to the CD equation. At large depths, the data approaches the noise floor, and it is not possible to resolve the two-peak structure. Accordingly, we fit only the region where the data are sufficiently precise to make a meaningful comparison. In Fig. 8, we show the resulting coefficients $c$ and $D$ extracted from the disk data and from several other data sets discussed below. We will return to this figure below.
B. Response for rectangular lattices of disks

In the experiments of this subsection, we consider how packing structure and friction affect the average responses for rectangular lattices. The motivation for such studies is the following. In a granular system, force is transmitted through the network of contacts. A seemingly simple and highly ordered geometric packing can have a highly complex contact network. For example, in a triangular packing of ideal disks, each disk except those at boundaries may have as many as six contact points, while the conditions of force and torque balance (in 2D) require two contacts located below the center of gravity of each disk. The first case is sometimes referred to as hyperstatic equilibrium [18]. In real packings, it is often possible that each particle (even a particle with low friction) can randomly lose several contacts without destroying the stability of the lattice. Conversely, even in a packing of frictionless particles, there can be a substantial range of forces at contacts, with a high degree of randomness. If friction is introduced, non-normal forces are allowed, the number of degrees of freedom increases, and effectively, the conditions for stability are relaxed even further. Hence, a seemingly “ordered” system from the point of view of geometrical packing can contain disorder in the contact forces, due to the shape and size variation of disks and, more importantly, to the existence of friction.

To decrease the randomness of forces at contacts, we constructed rectangular lattices. In this case, the number of contacts between disks was minimal and the contact network was well defined in the sense that the force at every contact was nonzero. We emphasis, however, that randomness in contact forces still exists, due to friction.

A typical rectangular monodisperse packing is shown in Fig. 10. To construct this packing, disks on the bottom layer were supported by a template consisting of equally spaced grooves with a center-to-center spacing of 1.27 disk diameters. The system size is ~ 85 wide and ~ 15 particles high. Since a rectangular lattice has less contacts than a triangular lattice, it is also less stable than a triangular lattice. Consequently, it was more difficult to build tall layers, and once built, a layer could not support as large forces as in
the case of triangular packings. Therefore, in this set of experiments, we used a 20 gram force for the probe.

We note that the naturally occurring surfaces of the disks were relatively frictional, with a static friction coefficient $\mu$ close to 0.94. In separate experiments, we wrapped each disk with Teflon tape, thus reducing the friction coefficients to about $\mu = 0.48$. This allowed us to investigate the role of disorder associated with friction in the force response.

In Fig. 11a-b, we show the grey-scale average response pictures for the rectangular lattice systems with large and small friction coefficients, respectively. In Fig. 12, we show quantitative data at several depths for both systems. In both cases, the responses propagate along the lattice directions. The measured value of the angle between the two propagation directions is $\sim 79^\circ$, which corresponds well with the rectangular lattice structure. This is qualitatively similar to the results for a triangular lattice, where the angle between the two preferred directions was 60 degrees. For both the hexagonal packing and the more frictional square packing, the peaks in the response broadened relatively rapidly with depth. When the friction was decreased (Teflon wrapped particles) the peaks remained significantly sharper with depth, as shown in Fig. 13. Here, the width $w$ is obtained by fitting a Gaussian curve $F = F_0 e^{-\left(\frac{x-x_0}{a} \right)^2}$ to each peak at a given depth. Note that the smallest peak width is one particle diameter due to our averaging procedure. These data suggest that in a perfectly ordered and frictionless system, the force response would be perfectly sharp force chains, as in the model of Bouchaud et al.

To close this section, we return to Fig. 8. In total, this figure shows results of fits to the convection-diffusion model for five different systems. These systems include two rectangular lattices with friction coefficients $\mu = 0.48$ and $\mu = 0.94$, a hexagonal packing of monodisperse disks, and the two randomly packing bidisperse disk systems with $A = 0.993$ and 0.988. We find that the coefficient $c$ decreases and the coefficient $D$ increases with disorder, and that $c$ decrease roughly linearly with increasing $D$, which is consistent with the model of Bouchaud et al.
C. Non-normal force responses

We consider the vector character of force propagation in this section, namely the response to forces applied at arbitrary angles to the surface. We first consider the response to a non-normal force in a disordered system, one consisting of pentagonal particles. We then consider the corresponding problem for an ordered system, in this case monodisperse disks in a triangular packing.

For the case of a disordered system, such as a packing of pentagonal particles, an applied normal force produces a response that resembles that of an elastic solid. In order to provide a useful background for the non-normal case, we consider what happens in the corresponding situation of a non-normal force applied to a semi-infinite elastic plate. When a force is applied to such a plate of thickness $t$ at an angle $\theta$ with respect to the horizontal direction, as depicted in Fig. 14, the stress tensor components are [19]:

$$
\sigma_{rr} = \frac{2Ft}{\pi r} \cos \phi, \quad \sigma_{\phi \phi} = \sigma_{\phi \phi} = 0, \quad (6)
$$

where the angle $\phi$ is measured from the direction of the applied force, $r$ is the distance from the point under consideration to the point of contact. The dashed circles shown in Fig. 14 represent loci of equal stress. When the direction of applied force changes, these equal-stress lines remain the same with respect to the direction of the force, except for those points that lie outside of the material.

In order to determine what happens in granular media, we measured responses to forces applied over a range of angles for a system of pentagons. As in the previous measurements, particles were placed in a vertical plane, and forces were applied on a single grain at an angle $\theta$ with respect to the horizontal direction. Specifically, a force of 50g was applied to the surface at angles, 90°, 60°, 45° and 30° with respect to the horizontal. The line of force was chosen so that, as much as possible, it passed through the center of gravity of the grain. In other respects, the procedure and analysis were the same way as we described previously.

In Fig. 15, we show the grey-scale representation of the average response pictures for this system. In general, the force responses are centered along the direction of the applied forces.
and they are similar to a rotated version of the response when a vertical force is applied. This is more evident in Fig. 16. To obtain this figure, we first rotated the responses to the vertical direction and then obtained the force responses along a series of horizontal lines at depths, z, measured from the source. We then rescaled these responses as follows: we normalized the x coordinate by the value of the depth, z, and we multiplied the stresses by z.

For comparison, we plot the elastic plate solutions based on Eq. 6 in these figures. Fig. 16a is a confirmation that the response to a vertical force is consistent with that of an elastic material, i.e., the widths of response vary linearly with the depth. Fig. 16b, c and d show that the mean responses to forces at other angles in a pentagonal system can be roughly described by an elastic-like model. The deviations from the elastic solution on the $x < 0$ side in b, c and d may be attributable to the fact that there are no tensile forces in a granular material.

Next, we consider the results for a triangular packing of monodisperse disks. In Fig. 17, we show the grey-level average response pictures for systems where a force of 50g was applied to the surface at angles: 90°, 75°, 60°, 45°, 30° and 15°, with respect to the horizontal. For the $\theta = 90°$ case, the mean response is along the two lattice directions closest to the applied force direction, and left-right symmetry is preserved. For the $\theta = 75°$ case, the left-right symmetry is broken, but the response still involves the same two lattice directions. However, the response along the left lattice direction (i.e. most closely aligned along the applied force direction) is strong, and the response along the right lattice direction is relatively weak. For the $\theta = 60°$ case, where the force is applied along only one of the inherent lattice directions, we see that the response is only along that direction. For each of the $\theta = 45°, 30°$ and 15° cases, part of the average responses is aligned along the left lattice direction, and part is aligned along a new direction that is $\sim 62.5°$ clock-wise from the vertical direction. This is illustrated quantitatively in Fig. 18. To obtain this figure, we first partitioned the responses into small angular bins, 5° in width, and we then calculated the integral over the radial direction of the responses in those bins. Thus, these curves show the total strength of the
response in a given direction. The dominant lattice directions are $-30^\circ$ and $+30^\circ$. The other direction $\phi = -62.5$ that appears in this figure is associated with the next-nearest neighbor lattice direction.

**D. Responses of a system with shear deformation**

In this section, we consider the characterization of stress chain orientation and length in a system that has been subjected to a modest amount of uniform shear; in essence, this is a probe of the nature of texture. We then explore how texture affects force responses in a granular media.

We created a texture by applying nearly uniform shear. To do so, we constructed an experimental setup that is shown in Fig. 19(a). The particles, in this case pentagons, rested on a flat horizontal surface consisting of Plexiglas, and they were confined by Plexiglas walls. Two parallel boundaries were hinged at one corner so that it was possible to shear the system. The other two boundaries remained parallel during the shearing process. One of them remained fixed relative to the Plexiglas bottom plate, and the other, which was opposite the hinges, was guided so as to keep constant the distance between the opposite parallel boundary. Hence the available area to the disks also remained constant. We applied controlled amounts of shear to this system by slowly displacing the upper left corner of the boundary by a measured amount. The system size was about ~ 47 cm $\times$ ~ 22 cm. The pentagons were ~ 6.3 mm on an edge. The number of particles in the system was 1167 and the packing fraction was 0.795. By taking images without polarizers, we were able to obtain particle positions, including their centers of mass, as shown in Fig. 19(a).

The experimental procedure was the following: (1) the left edge was pushed continuously until it reached a given angle, $\phi$, with respect to the normal direction. A typical series of stress patterns as $\phi$ was increased is shown in Fig. 19(b). (2) Then a small local force perpendicular to the top edge was applied on a particle at the top boundary and the response image was recorded. (3) The force was removed and the background image was recorded.
(4) We then followed the general image processing procedure as described above to obtain the mean response. (5) We also characterized the orientation and length of stress chains which are typified by Fig. 19(b), through the use of two point spatial correlation functions for the stress.

We show in Fig. 19b the impact of applying a small shear deformation on the stress chains. This set of images follows the course of a deformation ending with \( \phi = 4.8^\circ \). An obvious result of this deformation is that the stress chains tend to align in a direction that opposes the deformation. A similar effect was observed for the orientation of stress chains in 2D Couette shear [22]. The stress chain orientation tended to saturate following a small angular deformation, i.e. for \( \phi \gtrsim 5^\circ \), the typical stress chain angle did not significantly change.

An important issue concerns the spatial structure of the stress chains that are generated in response to shear. With images such as those shown in Fig. 19(b)c and d, where stress chains are well defined, we can characterize the stress chain orientation and chain length by calculating the spatial auto-correlation \( c(\mathbf{r}) \) for \( G^2 \) responses. We define \( R(\mathbf{x}) \equiv G^2(\mathbf{x}) \), so:

\[
    c(\mathbf{r}) = c(r, \theta) = \langle G^2(\mathbf{x})G^2(\mathbf{x} + \mathbf{r}) \rangle,
\]

where the brackets denote a volume average over spatial coordinates \( \mathbf{x} \). The value of the \( c(r, \theta) \) carries information about how far away different parts of the micro-structure still "feel" each other. Particularly in the present case, the stress correlation function provides information about the anisotropic features of the system. The actual calculation of the correlation function is performed in the frequency domain using FFT (Fast Fourier Transform) techniques, since the computation in the space domain is cumbersome when the image size is large, e.g. 512 \( \times \) 512 pixels.

Fig. 22 shows such a spatial auto-correlation function \( c(r, \theta) \) represented in two different representations. These data are obtained by averaging 50 realizations. Clearly, and perhaps not surprisingly, these images show that the correlation function along the stress chain directions is much stronger than along the perpendicular direction, even though the stress
chain directions span a finite range of angles. The strongest direction for $c(r, \theta)$ is about 45° from the vertical. Fig. 23 shows the correlation function evaluated along this direction and the direction perpendicular to it. Along the perpendicular direction, the correlation is almost a δ function, dropping rapidly to a value close to zero over a distance of about 1 grain diameter. However, along the parallel direction, the correlation decreases like a power law with an exponent of −0.81, showing long range order over the size of the system.

The shear-induced anisotropic contact network had a significant impact on force propagation. When a vertical force was applied, the force tended to propagate along or toward the stress chains. Thus, on average, the force propagation direction deviated from the vertical direction, as shown in Fig. 20. This is in contrast to the response for an isotropic system of pentagons. In Fig. 21a, we give data for the response at different depths. In Fig. 21b we plot the same responses but rescale the x coordinates with depth $z$. In this latter figure, all peaks of the responses at different depths are roughly located around $x/z = 0.5$, which is about 25 ± 4° from the vertical. In order to shed light on this particular angle, we note, some other relevant angles in the system. These include: (1) the angle of internal friction for pentagons is 42.6 ± 1.9°, which is obtained by making a pile of pentagons and then measuring the angle of repose; (2) the angle of friction between the wall and the pentagons is about 29°, which is obtained by placing a particle on a slope, tilting the slope and then recording the angle when the particle starts to slip.

**IV. CONCLUSIONS**

We have measured the force response of 2D granular systems to local perturbations under various conditions. There are large variations from realization to realization, and we consider averages over many observations. We have obtained ensemble-averaged responses for four types of systems: bidisperse systems with different amount of disorder, systems packed as a rectangular lattice, systems with forces applied at an arbitrary angle at the surface, and systems with shear deformation, hence with textured/anisotropic features. We
find that disorder, packing structure, friction and textures affect the average force response in a granular system significantly. Specifically, we have found that: 1) in bidisperse systems, when the amount of disorder is increased by adjusting the size and number ratio of large and small disks, the average response changes from a response with two-peak feature to a one-peak response. 2) In a rectangular lattice system, the force propagates along the lattice direction and when the friction between particles is decreased, the mean response becomes sharper. 3) When a force of arbitrary direction is applied at the surface, for a disordered packing of pentagonal particles, the mean response can be roughly described by an elastic solution. For a hexagonal packing of disks, the mean response propagates along lattice directions and the anisotropy in the response is strong. 4) In a system with shear deformation, the stress chains tend to be oriented along such a direction as to resist the relative motion of boundaries, and the resulting average force response to a vertical local force tends to propagate along or toward these existing stress chain directions.

In closing this section, we note that according to conventional wisdom, there exists on some scale a large enough volume, called the representative elementary volume, or REV, such that the average properties of the material give an accurate description on scales greater than the REV. The expectation would be that the response to a small force could be obtained at the micro scale, and related to macroscopic continuum models. In the context of these experiments, that situation does not seem to apply. The range of the mean response, as characterized for example by the diffusion coefficient, $D$, is sufficiently short range, that it is not reasonable to invoke an average over a volume that consists of many grains. The situation is complicated by the fact that large amount of fluctuations from realization to realization.

These results help identify the important factors that affect force propagation in granular media and thus raise the need to incorporate these factors into models. There are a number of other interesting problems concerning force propagation in granular systems. These include the force response due to distributed load and force response when plastic deforations occur. We will address these issue in future work.
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V. APPENDIX

There are two stress equations of interest here, one is the two branch Convection-Diffusion equation (CD) adapted from Ref. [14] which is relevant when the two peak structure is prominent, the other is the elastic stress equation when the disorder is strong [4]. The CD equation takes the form of:

\[ \mathcal{O}^+ \mathcal{O}^- \sigma_{zz}(x, z) = 0 \],

(8)

where \( \mathcal{O}^\pm = \partial_x - D \partial_{xx} \pm c \partial_z \), \( x \) and \( z \) are horizontal and downward coordinates, and \( c \) and \( D \) are two parameters in analogy to a dimensionless velocity and a diffusion coefficient. The solution to this equation for a given \( F \delta(x, 0) \) initial condition is:

\[ \sigma_{zz}(x, z) = \frac{F}{2} \left( \frac{1}{\sqrt{4 \pi D z}} e^{-\frac{(x-cz)^2}{4Dz}} + \frac{1}{\sqrt{4 \pi D z}} e^{-\frac{(x+cz)^2}{4Dz}} \right) \]

(9)

where \( F \) is the magnitude of the downward force. For a given force, the stress at point \( (x, z) \) is always proportional to \( F \). In order to test the linearity, for an ordered disk packing, we only sampled points along the directions of \( x \pm cz \) where the peaks of responses are located. The stresses at those points are only determined by the depth \( z \) since \( \sigma_{zz}(x \pm cz) = \frac{F}{4\sqrt{\pi D z}} e^{-\frac{cz^2}{4\sqrt{\pi D z}}} \).

In Fig. 5(a), we plot the measured stress versus applied force at different depths. We find that when the applied force is below about 0.5 N, there is a good linearity within the error bars, linear region, because of the deformability of the grains we used (Poisson ratio is 0.5), besides the forces propagate along two lattice direction, it also tends to be is In our experiments, in order to increase the signal to noise ratio, we usually chose a working force
close to the upper bound of this linear region. For the disordered responses, we use the elastic solution:

$$\sigma_{zz}(x, z) = \frac{2F}{z\pi} \frac{1}{[(x/z)^2 + 1]^2}. \quad (10)$$

Similar to the idea above for ordered packing, we here only sampled the peak points where \(x=0\), i.e., \(\sigma_{zz}(0, z) = \frac{2F}{z\pi}\). Additionally, if we multiply the measured stress by the corresponding depth \(z\), we expect the result to be on the same straight line for different depths. In Fig. 5(b), we plot the measured stress multiplied by the corresponding depth against the applied force. As expected, we do see those lines for different depths to collapse on the same line, however, this linearity only holds when the applied force is less than 0.5 N.
REFERENCES


FIGURES

FIG. 1. a) Schematic view of the Circular Polarimeter setup. In our experiment the linear polarizer and quarter wave plates are combined into a single sheet. b) Fringe pattern of a disk under diametrical compression. c) Schematic drawing of pixel scale intensity and directions used to calculate $G^2$.

FIG. 2. Comparison of fringe pattern resulting from analytical calculation (left) with experimental data (right) of diametrical compression of one disk at 1.8 Newtons.
FIG. 3. a). Top section: Hydrostatic pressure due to gravitational force alone versus depth determined from $G^2$. The expected slopes of the stress-height curves are calculated from the known packing fraction ($\gamma = 0.91$ for disks; $\gamma \approx 0.75$ for pentagons). Inset shows the multi-particle $G^2$ calibration by applying known loads to the upper surface of the layer. b). Bottom section: Schematic diagram of the experimental apparatus. The drawing is not to scale and the aspect ratio of the system is about 3 in order to avoid the boundary effect. F is the applied vertical load.
FIG. 4. a). The grey-level image representing the standard deviation of $G^2$ for 50 trials of a 50 g point force on a hexagonal packing of disks. b). The distributions of the ratio of standard deviation to the mean $G^2$ for different systems.
FIG. 5. Linearity test: a). The measured stress vs. applied force for peaks at different depths in a system composed of disks. b). The measured stress multiplying the corresponding depth vs. applied force for different depths in a pentagonal system.

FIG. 6. Mean response for 50 trials of a 50 g point force for bidisperse systems with different amount of disorder, (a) \( \mathcal{A} = 0.993 \) (b) \( \mathcal{A} = 0.988 \) and (c) \( \mathcal{A} = 0.965 \). The size of each image is 300 \( \times \) 400 pixels (about 18.0 \( \times \) 13.5 cm). The free surface of the granular material is 20 lines from the top of the image.
FIG. 7. Photoelastic response $\langle G^2 \rangle$ to a point force vs. horizontal distance $x$ at various depths $z$ from the source for bidisperse systems with a) $A = 0.993$, b) $A = 0.988$ and c) $A = 0.965$, respectively.
FIG. 8. The coefficients of \( c \) and \( D \) extracted by fitting the mean responses from different systems to the CD equation. Each point in the graph corresponds to a different system. See the text for more details.

FIG. 9. Particle-particle autocorrelation function for a) hexagonal packed monodisperse disks, b), c) and d) for three bidisperse systems with a) \( A = 0.993 \), b) \( A = 0.988 \) and c) \( A = 0.965 \), respectively. The distances are normalized by the diameter of the smaller particles in each system. In a), the dashed line is calculated from a generated perfect triangular lattice and the solid line is from the experiments. In b), c) and d), \( g_{11} \) and \( g_{22} \) are correlation functions for same species of particles and \( g_{12} \) are for different species.
FIG. 10. A rectangular packing of disks with a lattice constant $a = 1.27d$, and $d$ is the disk diameter.

FIG. 11. Mean response for 50 trials of a 20 g point force for rectangular packing of disks. The coefficients of friction, $\mu$, between particles are a) 0.94 and b) 0.48.

FIG. 12. Photoelastic response $G^2$ to a point force, vs. horizontal distance, $x$, at various depths, $z$, from the source for rectangular packings of disks with a coefficient of friction (a) $\mu = 0.94$, and (b) $\mu = 0.48$. 

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FIG. 13. Width of peaks v.s. depth for rectangular packings of disks with a coefficient of friction (a). $\mu = 0.94$, and (b) $\mu = 0.48$.

FIG. 14. Schematic drawing of a force at an angle $\theta$ with respect to the horizontal is applied on the surface of a semi-infinite plate. $r$ is measured from the point of contact and the $\phi$ is measured counter-clock-wise from the direction of the force.
FIG. 15. Mean response for a system consisting of pentagonal particles where a force of 50g was applied on the surface at angles: (a) 90°, (b) 60°, (c) 45° and (d) 30°, with respect to the horizontal.

FIG. 16. Rescaled mean response at various depths from the source versus rescaled horizontal distance for a system consisting of pentagonal particles where a force of 50g was applied on the surface at angles: (a) 90°, (b) 60°, (c) 45° and (d) 30°, with respect to the horizontal.
FIG. 17. Mean response for a triangular packing of disks where a force of 50g was applied on the surface at angles: (a) 90°, (b) 75°, (c) 60°, (d) 45°, (e) 30° and (e) 15°, with respect to the horizontal.

FIG. 18. Relative strength of the response as a function of $\phi$ for different $\theta'$s. The definition of $\phi$ is illustrated in the inset.
FIG. 19. (a). A schematic drawing of the 2D shearing cell with real image overlayed on top of it. The particles are placed in a horizontal plane and their centers were identified by software. The left edge is pushed to generate a controlled shear deformation $\phi$. The local force $F$ perturbs the system in the direction perpendicular to the top edge. Note that the shearing cell is constructed in such a way so that the top edge always stays at the same level. (b). Image series showing stress chain patterns corresponding to different amount of shear deformation. The shear deformation increases with the image number. The shear deformation in image (d) is about 5 degrees.
FIG. 20. Mean response for 50 trials of a 50 g point force for a pentagonal system with a shear deformation of 4.7°. The force direction is perpendicular to the top edge.

FIG. 21. (a) More quantitative data for the averaged photoelastic response: pressure v.s. horizontal distance, x, at various depths, z, measured from the source. (b) The same data as (a), but the x coordinates are rescaled with depths h
FIG. 22. Spatial auto-correlation function $c(r, \theta)$ for stress chain patterns, typified by Fig. 19(b,d), in a square shear cell. The final correlation is an average of 50 images. (a) 3D representation of the $c(r, \theta)$. (b) 2D representation of the $c(r, \theta)$. The image size is $512 \times 512$ pixels, cropped from the original image which is $640 \times 480$ pixels and padded with the mean intensity, for computation efficiency. These images show that the correlation along the stress chain direction is strong.

FIG. 23. Spatial auto-correlation functions along $c(r, \theta = 135)$ along the direction parallel to the stress chain direction and $c(r, \theta = 45)$ along the direction perpendicular to the stress chain direction. $\theta$ is measured from the right horizontal direction. (a) shows correlation functions in a x-y plot and (b) shows correlation functions in a log-log plot. In (b), the correlation function parallel to the stress chain direction can be fitted with a power law decay: $c(r, \theta) \sim r^{-\gamma}$, where $\gamma$ is 0.78, showing a persistent long range order.